

ENTRY TIMES DISTRIBUTION FOR DYNAMICAL BALLS ON METRIC SPACES

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ABSTRACT. We show that the entry and return times for dynamic balls (Bowen balls) is exponential for systems that have an α -mixing invariant measure with certain regularities. We also show that systems modeled by Young's tower has exponential hitting time distribution for dynamical balls

1. INTRODUCTION

In this paper the distribution of entry and return times are studied for continuous maps on metric spaces. There are a great many results on the distribution of return times to cylinder sets for instance from 1990 by Pitskel [20] (see also [7]) for Axiom A maps using Markov partitions and also by Hirata. For ψ -mixing maps, Galves and Schmitt [9] have introduced a method to obtain the limiting distribution for the first entry time to cylinder sets. Abadi then generalised that approach to ϕ and also α -mixing measures. The nature of the return set however is critical for the longtime statistics of return and Lacroix and Kupka [15, 14] have given examples where for an ergodic system any return time distribution can be realised by taking a limit along a suitably chosen sequence of return sets. For entry and returns to balls, Pitskel's result shows using an approximation argument that for Axiom A maps on the two-dimensional torus return times are Poisson distributed. Recently, Chazottes and Collet [6] showed a similar result for attractors in the case of exponentially decaying correlations. This was in [13] extended to polynomial decay of correlations where the error terms are logarithmic. A similar result without error terms and requiring sufficient regularity of the invariant measure was proven in [19] (see also [10]). In this paper we consider another class of entry sets, namely dynamic balls on metric spaces. It has been shown elsewhere that dynamic balls exhibit good limiting statistic and in particular have the equipartition property which for partitions is associated with the theorem of Shannon-McMillan-Breiman [16]. Similarly, a theorem of Ornstein

Date: November 3, 2014.

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and Weiss [17, 18] has a counterpart for metric balls [21]: It was shown that the exponential growth rate of the re-entry times is equal to the metric entropy.

In this paper we first study the distribution of the first entry time assuming that there is a generating partition which is α -mixing. This requirement is satisfied by a large number of systems, in particular by those which allow Young's tower construction. This is shown in section 5. In section 3 we deduce the first return time distribution.

2. MAIN RESULTS

Let (X, d) be a compact metric space, $T : X \rightarrow X$ and (X, T, μ) be a continuous transformation and μ a T -invariant Borel probability measure on X . For a set $B \subset X$ denote by $\tau_B(y)$ the first time when the orbit of y enters B , i.e. $\tau_B(y) = \min\{j > 0 : T^j y \in B\} \in \mathbb{N} \cup \{\infty\}$. The first return time $\tau_B|_B$ is almost surely finite by Poincaré's recurrence theorem and satisfies $\int_B \tau_B d\mu = 1$ by Kac's theorem if μ is ergodic and $\mu(B) > 0$. A large number of results on the limiting distribution have been proven in the case when B are cylinder sets, most notably by Galves and Schmitt [9] for ψ -mixing measures where they introduced a method which later was in particular by Abadi [1, 2] extended to the first entries and returns for ϕ -mixing and α -mixing measures.

If T is a map on a metric space Ω with metric d , then the n th Bowen ball is given by $B_{\varepsilon, n}(x) = \{y \in \Omega : d(T^j x, T^j y) < \varepsilon, 0 \leq j < n\}$. Bowen balls are used to define the metric entropy and also the pressure for potentials (see e.g. [22]). Then in [5] the equivalent for the theorem of Shannon-McMillan-Breiman was proven for Bowen balls. Namely

$$h(\mu) = \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} |\log \mu(B_{\varepsilon, n}(x))|$$

for almost every x . In [21] (see also [8]) that for an ergodic T -invariant probability measure μ one has

$$\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} \log R_{\varepsilon, n}(x) = h(\mu)$$

almost everywhere, where $R_{\varepsilon, n}(x) = \tau_{B_{\varepsilon, n}(x)}(x)$ is the recurrence time to the Bowen ball (the limit in n is \limsup or \liminf). In this paper we want to address the distribution of the entry time function $\tau_{B_{\varepsilon, n}(x)}$ and of the higher order returns $\tau_{B_{\varepsilon, n}(x)}^j$ which are defined by $\tau_{B_{\varepsilon, n}(x)}^1 = \tau_{B_{\varepsilon, n}(x)}$ and recursively $\tau_{B_{\varepsilon, n}(x)}^{j+1} = \tau_{B_{\varepsilon, n}(x)} + \tau_{B_{\varepsilon, n}(x)}^j \circ T^{\tau_{B_{\varepsilon, n}(x)}}$.

In order to get meaningful results we have to assume some mixing property. We will require the measure to be α -mixing with respect to a finite or countably infinite generating partition. For that purpose let \mathcal{A} be a measurable partition of X which is generating and either finite or countably infinite. Denote by $\mathcal{A}^k = \bigvee_{j=0}^{k-1} T^{-j} \mathcal{A}$

its n -th join and write $\gamma_n = \text{diam}(\mathcal{A}^n)$ for its diameter. Clearly γ_n is a decreasing sequence. We shall require that the measure μ is α -mixing with respect to a such a partition \mathcal{A} , that is

$$|\mu(A \cap T^{-n-k}B) - \mu(A)\mu(B)| \leq \alpha(k)$$

for all $A \in \sigma(\mathcal{A}^n)$, $B \in \sigma(\bigcup_j \mathcal{A}^j)$, where $\alpha(k)$ is a decreasing function which converges to zero as $k \rightarrow \infty$.

In addition to a mixing property we will require some regularity of the measure. For $0 < \delta < \epsilon$ define the function

$$\varphi(\epsilon, \delta, x) = \frac{\mu(B(x, \epsilon + \delta)) - \mu(B(x, \epsilon - \delta))}{\mu(B(x, \epsilon))}$$

for $x \in X$. The function φ measures the proportion of the measure of the annulus $B(x, \epsilon + \delta) \setminus B(x, \epsilon - \delta)$ to the ball $B(x, \epsilon)$. This is needed in order to control the approximation of balls by cylinder sets below.

For $\epsilon > 0$ and $n \in \mathbb{N}$ we then define the (ϵ, n) -Bowen ball

$$B_{\epsilon, n}(x) = \{y : \sup_{0 \leq k \leq n-1} d(T^k x, T^k y) < \epsilon\}.$$

We now can formulate our main results.

Theorem 1. *Let μ an α -mixing T -invariant probability measure on Ω . Assume that there exist constants $0 < \gamma < 1$ and $\zeta, \kappa > 0$ such that $\text{diam}(\mathcal{A}^n) = \mathcal{O}(\gamma^n)$, $\alpha(n) = \mathcal{O}(n^{-(2+\kappa)})$ and*

$$\varphi(\epsilon, \delta, x) \leq \frac{C_\epsilon}{|\log \delta|^{3+\zeta}}$$

for some constant $C_\epsilon > 0$ that does not depend on x and all δ small enough.

Then there exists an $\omega > 0$ and a constant C_1 so that

$$\left| \mathbb{P}(\tau_{B_{\epsilon, n}(x)} > \frac{t}{\lambda_{B_{\epsilon, n}(x)} \mu(B_{\epsilon, n}(x))}) - e^{-t} \right| \leq C_1 \mu(B_{\epsilon, n}(x))^\omega.$$

For simplicity let us put $B = B_{\epsilon, n}(x)$. If, as in the next result, the measure has good regularity then we relax the condition on the decrease of the diameter of cylinders considerably.

Theorem 2. *Assume that there exist constants, $a, \kappa, \zeta > 0$ satisfying $a\zeta > 3$, such that $\text{diam}(\mathcal{A}^n) = \mathcal{O}(n^{-a})$, $\alpha(n) = \mathcal{O}(n^{-(2+\kappa)})$ and*

$$\varphi(\epsilon, \delta, x) \leq C_\epsilon \delta^\zeta$$

for some constant C_ϵ .

Then there exists an $\omega > 0$ and a constant C_2 so that

$$\left| \mathbb{P}(\tau_{B_{\epsilon, n}(x)} > \frac{t}{\lambda_{B_{\epsilon, n}(x)} \mu(B_{\epsilon, n}(x))}) - e^{-t} \right| \leq C_2 \mu(B_{\epsilon, n}(x))^\omega.$$

While the previous two theorems give us limiting results for the entry times distribution, the next two theorems establish equivalent results for the return times.

For all set $A \subset \Omega$ define the *period* of A by $\tau(A) = \min\{k > 0 : T^{-k}A \cap A \neq \emptyset\}$ and put for any $\Delta < 1/\mu(A)$

$$a_A = \mathbb{P}_A(\tau_A > \tau(A) + \Delta).$$

In our setting we will choose $N(n) \leq \Delta \leq 1/\mu(\tilde{B}_{\epsilon,n}(x))$, where $N(n)$ will be determined later. Again we write for simplicity $\tilde{B} = \tilde{B}_{\epsilon,n}(x)$ and $B = B_{\epsilon,n}(x)$.

Theorem 3. *Let μ an α -mixing. Assume that there is a $\zeta > 0$ so that*

$$\varphi(\epsilon, \delta, x) \leq \frac{C_\epsilon}{|\log \delta|^{5+\zeta}}$$

for some constant C_ϵ . The remaining conditions are as in Theorem 1.

Then there exists an $\omega > 0$ so that

$$\left| \mathbb{P}_B(\tau_B > \frac{t}{\lambda_B \mu(B)}) - a_B e^{-t} \right| \leq C_3 \mu(B)^\omega$$

for some constant C_3 and a parameter λ_B which is bounded as in Lemma 4.

Theorem 4. *Assume there are $a, \kappa, \zeta > 0$ satisfying $a\zeta > 5$, such that $\text{diam}(\mathcal{A}^n) = \mathcal{O}(n^{-a})$, $\alpha(n) = \mathcal{O}(n^{-(2+\kappa)})$ and*

$$\varphi(\epsilon, \delta, x) \leq C_\epsilon \delta^\zeta.$$

The remaining conditions are as in Theorem 2.

Then there exists an $\omega > 0$ so that

$$\left| \mathbb{P}_B(\tau_B > \frac{t}{\lambda_B \mu(B)}) - a_B e^{-t} \right| \leq C_4 \mu(B)^\omega$$

for some constant C_4 .

The next result will be our principal technical result on which all the other theorems are based. For that purpose let $N(n)$ be an increasing sequence. We want to approximate the Bowen balls $B_{\epsilon,n}(x)$ by a unions on $N(n)$ -cylinders from the inside. For this purpose put

$$\tilde{B}_{\epsilon,n}(x) = \bigcup_{A^{N(n)} \in \mathcal{A}^{N(n)}, A^{N(n)} \subset B_{\epsilon,n}(x)} A^{N(n)}$$

which is the largest union of all $N(n)$ -cylinders contained in $B_{\epsilon,n}(x)$. The following is our main result.

Theorem 5. *Let μ be an α -mixing T -invariant probability measure on Ω . Assume there exist $\epsilon_0 > 0$ and an increasing sequence $\{N(n)\}_{n=1}^\infty$ satisfying $n < N(n) < \frac{1}{4}\mu(B_{\epsilon,n}(x))^{-1}$ such that*

$$(1) \quad \varphi(\epsilon, \gamma_{N(n)-k}, T^k x) \leq \vartheta_n(\epsilon) \cdot \frac{\mu(B_{\epsilon,n}(x))}{ns}$$

for all $\epsilon < \epsilon_0$, $x \in X$, $0 \leq k \leq n-1$, where $s = \alpha^{-1}(C'\mu(\tilde{B})) + N(n)$ for some $0 < C' < 1$ and $\vartheta_n(\epsilon) \rightarrow 0$ as $n \rightarrow \infty$ for every ϵ .

Then there exist $\lambda_{B_{\epsilon,n}(x)}$ with $\frac{C}{s} < \lambda_{B_{\epsilon,n}(x)} < 2$ and constants C_5, C_6 such that

$$\begin{aligned} & \left| \mathbb{P}(\tau_{B_{\epsilon,n}(x)} > \frac{t}{\lambda_{B_{\epsilon,n}(x)}\mu(B_{\epsilon,n}(x))}) - e^{-t} \right| \\ & \leq \vartheta_n(\epsilon) \frac{t}{s\lambda_{B_{\epsilon,n}(x)}} + 2f\mu(B_{\epsilon,n}(x)) + C_5 \frac{sN(n)}{f} + C_6 s \frac{\alpha(N(n))}{f\mu(\tilde{B})} \end{aligned}$$

for all $f \in (2N(n), \frac{1}{2}\mu(B_{\epsilon,n}(x))^{-1})$.

3. FIRST HITTING TIME DISTRIBUTION FOR BOWEN BALLS

In this section we will prove Theorems 1 and 2, but first we state several lemmata, the first one of which is evident.

Lemma 1. *For all n and x we have $\sum_{k=1}^{N(n)} \mu(\tilde{B}_{\epsilon,n}(x) \cap T^{-k}\tilde{B}_{\epsilon,n}(x)) \leq N(n)\mu(\tilde{B}_{\epsilon,n}(x))$ and $\mathbb{P}(\tau_{\tilde{B}_{\epsilon,n}(x)} \leq t) \leq t\mu(\tilde{B}_{\epsilon,n}(x))$.*

To simplify notation, we fix ϵ and n for a moment and write $B = B_{\epsilon,n}(x)$ and $\tilde{B} = \tilde{B}_{\epsilon,n}(x)$.

Lemma 2. *For all Δ, f such that $f \geq \Delta > N(n)$ and $g \in \mathbb{N}$ we have*

$$|\mathbb{P}(\tau_{\tilde{B}} > f+g) - \mathbb{P}(\tau_{\tilde{B}} > g)\mathbb{P}(\tau_{\tilde{B}} > f)| \leq 2\Delta\mu(\tilde{B}_{\epsilon,n}(x)) + \alpha(\Delta - N(n))$$

Proof. We proceed in the traditional way splitting the difference into three parts:

$$\begin{aligned} & |\mathbb{P}(\tau_{\tilde{B}} > g+f) - \mathbb{P}(\tau_{\tilde{B}} > g)\mathbb{P}(\tau_{\tilde{B}} > f)| \\ & \leq |\mathbb{P}(\tau_{\tilde{B}} > g+f) - \mathbb{P}(\tau_{\tilde{B}} > g \cap \tau_{\tilde{B}} \circ T^{g+\Delta} > f-\Delta)| \\ & \quad + |\mathbb{P}(\tau_{\tilde{B}} > g \cap \tau_{\tilde{B}} \circ T^{g+\Delta} > f-\Delta) - \mathbb{P}(\tau_{\tilde{B}} > g)\mathbb{P}(\tau_{\tilde{B}} > f-\Delta)| \\ & \quad + |\mathbb{P}(\tau_{\tilde{B}} > g)\mathbb{P}(\tau_{\tilde{B}} > f-\Delta) - \mathbb{P}(\tau_{\tilde{B}} > g)\mathbb{P}(\tau_{\tilde{B}} > f)| \\ & = I + II + III. \end{aligned}$$

The first term is estimated as follows

$$I = \mathbb{P}(\tau_{\tilde{B}} > g \cap \tau_{\tilde{B}} \circ T^{g+\Delta} > f-\Delta \cap \tau_{\tilde{B}} \circ T^g \leq \Delta) \leq \mathbb{P}(\tau_{\tilde{B}} \leq \Delta) \leq \Delta\mu(\tilde{B}).$$

Similarly for the third term

$$III = \mathbb{P}(\tau_{\tilde{B}} > g) \mathbb{P}(f - \Delta < \tau_{\tilde{B}} \leq f) \leq \mathbb{P}(\tau_{\tilde{B}} > g) \Delta\mu(\tilde{B}) \leq \Delta\mu(\tilde{B}).$$

For the second term we use the α -mixing property to obtain

$$II = |\mathbb{P}(\tau_{\tilde{B}} > g \cap \tau_{\tilde{B}} \circ T^{g+\Delta} > f - \Delta) - \mathbb{P}(\tau_{\tilde{B}} > g) \mathbb{P}(\tau_{\tilde{B}} > f - \Delta)| \leq \alpha(\Delta - N(n)).$$

The three parts combined now prove the lemma. \square

Let us now put $\theta = \theta(f) = -\log \mathbb{P}(\tau_{\tilde{B}} > f)$ where $f > 0$. We then have the following estimate.

Lemma 3. *Let $f > \Delta > N(n)$ then for all $k \geq 1$ we have*

$$|\mathbb{P}(\tau_{\tilde{B}} > kf) - e^{-\theta k}| \leq \frac{2\Delta\mu(\tilde{B}) + \alpha(\Delta - N(n))}{\mathbb{P}(\tau_{\tilde{B}} \leq f)}.$$

Proof. Clearly the lemma hold for $k = 1$ by definition of θ . For $k > 1$ we use induction. For the induction step we obtain:

$$\begin{aligned} & |\mathbb{P}(\tau_{\tilde{B}} > (k+1)f) - e^{-\theta(k+1)}| \\ & \leq |\mathbb{P}(\tau_{\tilde{B}} > (k+1)f) - \mathbb{P}(\tau_{\tilde{B}} > kf) \cdot e^{-\theta}| + |\mathbb{P}(\tau_{\tilde{B}} > kf) \cdot e^{-\theta} - e^{-\theta(k+1)}| \\ & \leq 2\Delta\mu(\tilde{B}) + \alpha(\Delta - N(n)) + e^{-\theta} |\mathbb{P}(\tau_{\tilde{B}} > kf) - e^{-\theta k}| \\ & \leq 2\Delta\mu(\tilde{B}) + \alpha(\Delta - N(n)) + e^{-\theta} (2\Delta\mu(\tilde{B}) + \alpha(\Delta - N(n))) \cdot (1 + e^{-\theta} + \dots + e^{-\theta(k-2)}) \\ & = (2\Delta\mu(\tilde{B}) + \alpha(\Delta - N(n))) \cdot (1 + e^{-\theta} + \dots + e^{-\theta(k-1)}). \end{aligned}$$

Hence

$$|\mathbb{P}(\tau_{\tilde{B}} > kf) - e^{-\theta k}| \leq (2\Delta\mu(\tilde{B}) + \alpha(\Delta - N(n))) \frac{1}{1 - e^{-\theta}}$$

for all $k \in \mathbb{N}$ and the lemma follows since $\frac{1}{1 - e^{-\theta}} = \frac{1}{\mathbb{P}(\tau_{\tilde{B}} \leq f)}$. \square

For subsets $B \subset X$ let us define

$$\lambda_{B,f} = \frac{-\log \mathbb{P}(\tau_B > f)}{f\mu(B)}.$$

For the approximations $\tilde{B}_{\epsilon,n}(x)$ we then obtain the following estimate.

Lemma 4. *Let $f \in \mathbb{N}$ be such that $f\mu(\tilde{B}_{\epsilon,n}(x)) \leq \frac{1}{2}$. Then there exist $C_7 > 0$ such that*

$$\frac{C_7}{s} \leq \lambda_{\tilde{B}_{\epsilon,n}(x),f} \leq 2$$

where, as before, $s = \alpha^{-1}(C'\mu(\tilde{B}_{\epsilon,n}(x))) + N(n)$ for some $0 < C' < 1$

Proof. We follow the proof in Galves-Schmitt [9] and Abadi [3]. To estimate $\lambda_{\tilde{B}_{\epsilon,n}(x),f}$ we use the simple estimate

$$\frac{\theta}{2} \leq 1 - e^{-\theta} \leq \theta$$

for all $\theta \in [0, 1]$. Let us write \tilde{B} for $\tilde{B}_{\epsilon,n}(x)$ and note that $\mathbb{P}\{\tau_{\tilde{B}} \leq f\} = 1 - e^{-\theta}$ as $\theta = -\log \mathbb{P}\{\tau_{\tilde{B}} > f\}$. By Lemma 1,

$$\lambda_{\tilde{B},f} = \frac{\theta}{f\mu(\tilde{B})} \leq \frac{2\mathbb{P}\{\tau_{\tilde{B}} \leq f\}}{f\mu(\tilde{B})} \leq 2$$

For the lower bound, notice that $\{\tau_{\tilde{B}} > f\} = \bigcap_{j=0}^{[f]} \left(T^{-js}(\tilde{B})\right)^c \subset \bigcap_{j=0}^{[\frac{f}{s}]} \left(T^{-js}(\tilde{B})\right)^c$. As in the proof of Lemma 3 we obtain by induction

$$\mu \left(\bigcap_{j=0}^{k+1} \left(T^{-js+1}(\tilde{B})\right)^c \right) \leq \mu \left(\bigcap_{j=0}^k \left(T^{-js+1}(\tilde{B})\right)^c \right) \mu(\tilde{B}^c) + \alpha(s - N(n))$$

which yields

$$\mu \left(\bigcap_{j=0}^{[\frac{f}{s}]} \left(T^{-js+1}(\tilde{B})\right)^c \right) \leq \mu(\tilde{B}^c)^{[\frac{f}{s}]} + \alpha(s - N(n)) \frac{1}{1 - \mu(\tilde{B}^c)}.$$

Consequently

$$\mathbb{P}(\tau_{\tilde{B}} > f) \leq (1 - \mu(\tilde{B}))^{f/s} + \alpha(s - N(n)) \frac{1 - (1 - \mu(\tilde{B}))^{f/s}}{\mu(\tilde{B})},$$

and therefore

$$\begin{aligned} \mathbb{P}(\tau_{\tilde{B}} \leq f) &\geq (1 - (1 - \mu(\tilde{B}))^{f/s}) \left(1 - \frac{\alpha(s - N(n))}{\mu(\tilde{B})}\right) \\ &\geq \frac{f}{s} \mu(\tilde{B}) \left(1 - \frac{\alpha(s - N(n))}{\mu(\tilde{B})}\right). \end{aligned}$$

Thus

$$\lambda_{\tilde{B},f} = \frac{\theta}{f\mu(\tilde{B})} \geq \frac{\mathbb{P}\{\tau_{\tilde{B}} \leq f\}}{f\mu(\tilde{B})} \geq \frac{1}{s} \left(1 - \frac{\alpha(s - N(n))}{\mu(\tilde{B})}\right).$$

In particular since $s = \alpha^{-1}(C'\mu(\tilde{B})) + N(n)$ we get $\lambda_{\tilde{B},f} \geq \frac{C_7}{s}$ and $\mathbb{P}(\tau_{\tilde{B}} \leq f) \geq \frac{C_7 f \mu(\tilde{B})}{s}$ for some constant C_7 . \square

Now let us define the ‘annulus’

$$\widetilde{\partial B}_{\epsilon,n}(x) = \bigcup_{A^{N(n)} \in \mathcal{A}^{N(n)}, A^{N(n)} \cap \partial B_{\epsilon,n}(x) \neq \emptyset} A^{N(n)}.$$

We then have that $B_{\epsilon,n}(x) \setminus \widetilde{B}_{\epsilon,n}(x) \subset \widetilde{\partial B}_{\epsilon,n}(x)$. We also have $\tau_{B_{\epsilon,n}(x)} \geq \tau_{\widetilde{B}_{\epsilon,n}(x)}$ since $\widetilde{B}_{\epsilon,n}(x) \subset B_{\epsilon,n}(x)$. The following lemma estimates the size of the annulus.

Lemma 5. *With the notation as above (and in particular with $s = \alpha^{-1}(C'\mu(\widetilde{B})) + N(n)$) we obtain*

$$\mu(\widetilde{\partial B}_{\epsilon,n}(x)) = \vartheta_n(\epsilon) \frac{\mu(B_{\epsilon,n}(x))}{s}.$$

Proof. Since T is continuous, $\partial B_{\epsilon,n}(x) \subset \bigcup_{k=0}^{n-1} T^{-k} \partial B(T^k x, \epsilon)$. Hence if $A^{N(n)} \cap \partial B_{\epsilon,n}(x) \neq \emptyset$ then we must have $A^{N(n)-k}(T^k y) \cap \partial B(T^k x, \epsilon) \neq \emptyset$ for some $0 \leq k \leq n-1$, $y \in A^{N(n)}$. Notice that $\text{diam}(A^{N(n)-k}(T^k y)) \leq \gamma_{N(n)-k}$, we have

$$\begin{aligned} \widetilde{\partial B}_{\epsilon,n}(x) &\subset \bigcup_{k=0}^{n-1} T^{-k}(B(\partial B(T^k x, \epsilon), \gamma_{N(n)-k})) \\ &\subset \bigcup T^{-k}(B(T^k x, \epsilon + \gamma_{N(n)-k}) \setminus B(T^k x, \epsilon - \gamma_{N(n)-k})), \end{aligned}$$

hence

$$\begin{aligned} \mu(\widetilde{\partial B}_{\epsilon,n}(x)) &\leq n \cdot \sup_{0 \leq k \leq n-1} \mu(B(T^k x, \epsilon + \gamma_{N(n)-k}) \setminus B(T^k x, \epsilon - \gamma_{N(n)-k})) \\ &= n \cdot \sup_{0 \leq k \leq n-1} \{\varphi(\epsilon, \gamma_{N(n)-k}, T^k x) \cdot \mu(B(T^k x, \epsilon))\} \\ &= \vartheta_n(\epsilon) \frac{\mu(B_{\epsilon,n}(x))}{s} \sup_{0 \leq k \leq n-1} \mu(B(T^k x, \epsilon)) \\ &= \vartheta_n(\epsilon) \frac{\mu(B_{\epsilon,n}(x))}{s}. \end{aligned}$$

In particular we have $\mu(B_{\epsilon,n}(x))/\mu(\widetilde{B}_{\epsilon,n}(x)) = \mathcal{O}(1)$. □

Lemma 6. *For $f \leq \frac{1}{2}\mu(\widetilde{B}_{\epsilon,n}(x))^{-1}$ one has*

$$\left| \mathbb{P}(\tau_{B_{\epsilon,n}(x)} > \frac{t}{\lambda_B \mu(B_{\epsilon,n}(x))}) - \mathbb{P}(\tau_{\widetilde{B}_{\epsilon,n}(x)} > \frac{t}{\lambda_B \mu(\widetilde{B}_{\epsilon,n}(x))}) \right| \leq 3 \frac{\vartheta_n(\epsilon)t}{s \lambda_{\widetilde{B},f}}$$

for all $t > 0$.

Proof. Let us write $\lambda_B = \lambda_{\tilde{B},f}$. Then

$$\begin{aligned} & \left| \mathbb{P}(\tau_B > \frac{t}{\lambda_B \mu(B)}) - \mathbb{P}(\tau_{\tilde{B}} > \frac{t}{\lambda_B \mu(\tilde{B})}) \right| \\ & \leq \left| \mathbb{P}(\tau_B > \frac{t}{\lambda_B \mu(B)}) - \mathbb{P}(\tau_B > \frac{t}{\lambda_B \mu(\tilde{B})}) \right| + \left| \mathbb{P}(\tau_B > \frac{t}{\lambda_B \mu(\tilde{B})}) - \mathbb{P}(\tau_{\tilde{B}} > \frac{t}{\lambda_B \mu(\tilde{B})}) \right| \\ & = I + II. \end{aligned}$$

We estimate the two terms separately.

For the term I first notice that $\tilde{B}_{\epsilon,n}(x) \subset B_{\epsilon,n}(x)$ which implies $\tau_{\tilde{B}_{\epsilon,n}(x)} \geq \tau_{B_{\epsilon,n}(x)}$ and $\frac{t}{\lambda_B \mu(B)} \leq \frac{t}{\lambda_B \mu(\tilde{B})}$. Therefore

$$\begin{aligned} I & \leq \mathbb{P} \left(\frac{t}{\lambda_B \mu(B)} \leq \tau_B \leq \frac{t}{\lambda_B \mu(\tilde{B})} \right) \\ & \leq \frac{t}{\lambda_B} \left(\frac{1}{\mu(\tilde{B})} - \frac{1}{\mu(B)} \right) \mu(B) \\ & \leq 2 \frac{\vartheta_n(\epsilon)t}{s\lambda_B} \end{aligned}$$

as $\frac{\mu(B)}{\mu(\tilde{B})} \leq 2$ and where we used that

$$(2) \quad \{\mathbf{r.diff}\} \frac{1}{\mu(\tilde{B})} - \frac{1}{\mu(B)} = \frac{\mu(B \setminus \tilde{B})}{\mu(B)\mu(\tilde{B})} \leq \frac{\mu(\partial \tilde{B}_{\epsilon,n}(x))}{\mu(B)\mu(\tilde{B})} \leq \vartheta_n(\epsilon) \frac{1}{s\mu(\tilde{B})}$$

by Lemma 5.

The term II we estimate as follows:

$$\begin{aligned} II & = \mathbb{P} \left(\left\{ \tau_B \leq \frac{t}{\lambda_B \mu(\tilde{B})} \right\} \cap \left\{ \tau_{\tilde{B}} > \frac{t}{\lambda_B \mu(\tilde{B})} \right\} \right) \\ & \leq \mathbb{P} \left(\tau_{\tilde{B}_{\epsilon,n}(x)} \leq \frac{t}{\lambda_B \mu(\tilde{B})} \right) \\ & \leq \frac{t}{\lambda_B} \frac{\mu(\partial \tilde{B}_{\epsilon,n}(x))}{\mu(\tilde{B})} \\ & = \frac{\vartheta_n(\epsilon)t}{s\lambda_B}, \end{aligned}$$

where in the last line we proceeded as for the term I above. Since by Lemma 4, $s\lambda_B > C_7/2$ the result follows. \square

Proof of Theorem 5. We have to estimate $|\mathbb{P}(\tau_{\tilde{B}_{\epsilon,n}(x)} > \frac{t}{\lambda_B \mu(\tilde{B}_{\epsilon,n}(x))}) - e^{-t}|$, where, as before, $\lambda_B = \lambda_{\tilde{B},f}$. We put $\Delta = 2N(n)$ and pick $f > \Delta = 2N(n)$ with $f \leq \frac{1}{2}\mu(\tilde{B}_{\epsilon,n}(x))^{-1}$. Then $t > 0$ can be written as $t = kf + r$ with $0 \leq r < f$ and k integer. Set $t' = t - r = kf$, then

$$\begin{aligned} & |\mathbb{P}(\tau_{\tilde{B}} > t) - e^{-\lambda_B \mu(\tilde{B})t}| \\ & \leq |\mathbb{P}(\tau_{\tilde{B}} > t) - \mathbb{P}(\tau_{\tilde{B}} > t')| + |\mathbb{P}(\tau_{\tilde{B}} > t') - e^{-\lambda_B \mu(\tilde{B})t'}| + |e^{-\lambda_B \mu(\tilde{B})t'} - e^{-\lambda_B \mu(\tilde{B})t}| \\ & = I + II + III. \end{aligned}$$

The first term is easily estimated by

$$I = \mathbb{P}(t' < \tau_{\tilde{B}} \leq t) \leq r\mu(\tilde{B}) < f\mu(\tilde{B}).$$

For the third term we use the mean value theorem according to which there exist $t_0 \in [\lambda_B \mu(\tilde{B})t', \lambda_B \mu(\tilde{B})t]$ such that

$$III = e^{-t_0} \lambda_B \mu(\tilde{B})r \leq 2f\mu(\tilde{B})$$

using Lemma 4 in the last estimate.

To the second term, II , we apply Lemma 3 and obtain

$$\begin{aligned} II & \leq \frac{2\Delta\mu(\tilde{B}) + \alpha(\Delta - N(n))}{\mathbb{P}(\tau_{\tilde{B}} \leq f)} \\ & \leq \frac{s(2\Delta\mu(\tilde{B}) + \alpha(\Delta - N(n)))}{Cf\mu(\tilde{B})} \\ & = C_5 \frac{sN(n)}{f} + C_6 s \frac{\alpha(N(n))}{f\mu(\tilde{B})}. \end{aligned}$$

All three estimates combined yield

$$|\mathbb{P}(\tau_{\tilde{B}_{\epsilon,n}(x)} > \frac{t}{\lambda_B \mu(\tilde{B}_{\epsilon,n}(x))}) - e^{-t}| \leq 2f\mu(B_{\epsilon,n}(x)) + C_5 \frac{sN(n)}{f} + C_6 s \frac{\alpha(N(n))}{f\mu(\tilde{B})}.$$

□

For the remaining results we will use approximations of Bowen balls by unions of cylinder sets of lengths $N(n)$. For that purpose let us establish the following notation. For some $0 < \eta < \frac{1}{2}, \beta \in (\eta, 1)$, which will be determined later, we define $N(n) = \mu(B_{\epsilon,n}(x))^{-\eta}$ (length of cylinders), $f = \mu(B)^{-\beta}$, $\tilde{B}_{\epsilon,n}(x) = \bigcup_{D \in \mathcal{A}^{N(n)}, D \subset B_{\epsilon,n}(x)} D$

(inner approximation) and

$$\lambda_{B_{\epsilon,n}(x)} = \frac{-\log \mathbb{P}(\tau_{\tilde{B}_{\epsilon,n}(x)} > f)}{f\mu(\tilde{B}_{\epsilon,n}(x))}.$$

Proof of Theorem 1. In order to apply Theorem 5 we first verify (1) with $\gamma_n = \mathcal{O}(\gamma^n)$. Fix some $0 < \eta < \frac{1}{2}$ and set $N(n) = \lceil \mu(B_{\epsilon,n}(x))^{-\eta} \rceil$. Then

$$\varphi(\epsilon, \gamma_{N(n)-k}, T^k x) \leq \frac{C_\epsilon}{|\log \gamma_{N(n)-k}|^{3+\zeta}} \leq \frac{C_\epsilon}{N(n)^{3+\zeta}} = C_\epsilon \mu(B)^{(3+\zeta)\eta}.$$

Since

$$(3) \quad \{\mathbf{s}\mathbf{s}\} \quad s = \alpha^{-1}(C' \mu(\tilde{B})) + N(n) \leq c_1 \mu(B)^{-\frac{1}{2+\kappa}} + \mu(B)^{-\eta}$$

for some constant c_1 , we have

$$\frac{ns\varphi(\epsilon, \gamma_{N(n)-k}, T^k x)}{\mu(B)} \leq \vartheta_n(\epsilon),$$

where

$$\vartheta_n(\epsilon) \leq C_\epsilon ns \mu(B)^{(3+\zeta)\eta-1} \leq c_2 n \mu(B)^{(3+\zeta)\eta-1} \left(\mu(B)^{-\frac{1}{2+\kappa}} + \mu(B)^{-\eta} \right),$$

which converges to 0 if $\eta > \eta_0 = \max\{\frac{1}{2+\zeta}, \frac{1}{3+\zeta} \frac{3+\kappa}{2+\kappa}\}$.

Applying Theorem 5 yields as $s\lambda_{B_{\epsilon,n}(x)} \geq C_7$:

$$|\mathbb{P}(\tau_{B_{\epsilon,n}(x)} > \frac{t}{\lambda_{B_{\epsilon,n}(x)} \mu(B_{\epsilon,n}(x))}) - e^{-t}| \leq \vartheta_n(\epsilon) t + 2f\mu(B_{\epsilon,n}(x)) + C_5 \frac{sN(n)}{f} + C_6 s \frac{\alpha(N(n))}{f\mu(\tilde{B})}.$$

Since $f = \mu(B)^{-\beta}$ for some $\beta < 1$, the second term on the RHS converges to 0. The last two terms then are bounded as follows:

$$C_5 \frac{sN(n)}{f} + C_6 s \frac{\alpha(N(n))}{f\mu(\tilde{B})} \leq c_3 \mu(B)^\beta \left(\mu(B)^{-\eta} + \mu(B)^{\eta(2+\kappa)-1} \right) \left(\mu(B)^{-\frac{1}{2+\kappa}} + \mu(B)^{-\eta} \right).$$

In order that all terms converge to 0 we need $\eta > \eta_0$ so that

$$\omega_1 = \beta - \max \left\{ \eta, \frac{1}{2+\kappa} \right\} - \max \{ \eta, 1 - \eta(2+\kappa) \}$$

is positive. This can be achieved by picking $\eta \in (\eta_1, \frac{1}{2})$, where $\eta_1 = \max\{\frac{1}{2+\zeta}, \frac{1}{2+\kappa}\}$ is less than $\frac{1}{2}$ (note $\eta_1 \geq \eta_0$). This also implies that $\omega_2 = (3+\zeta)\eta - \max\{\frac{1}{2+\kappa}, \eta\}$ is positive. Now put $\omega = \min\{\omega_1, \omega_2, 1 - \beta\}$. \square

Proof of Theorem 2. In order to apply Theorem 5 we verify that condition (1) holds:

$$\begin{aligned} \varphi(\epsilon, \gamma_{N(n)-n}, T^k x) &\leq C_\epsilon \gamma_{N(n)-k}^\zeta \\ &\leq c_1 (N(n) - k)^{-a\zeta} \\ &\leq c_2 N(n)^{-a\zeta} \end{aligned}$$

(for some c_1, c_2) and therefore $ns\varphi(\epsilon, \gamma_{N(n)-k}, T^k x)/\mu(B) \leq \vartheta_n(\epsilon)$, where

$$\vartheta_n(\epsilon) \leq c_3 ns\mu(B)^{a\zeta\eta-1} \leq c_4 n\mu(B)^{a\zeta\eta-1} \left(\mu(B)^{-\frac{1}{2+\kappa}} + \mu(B)^{-\eta} \right).$$

The last expression converges (exponentially fast) to zero if $a\zeta\eta - 1 - \max\{\frac{1}{2+\kappa}, \eta\}$ is positive which can be achieved by picking $\eta < \frac{1}{2}$ close enough to $\frac{1}{2}$.

The remainder of the proof is identical to the proof of Theorem 1. \square

4. FIRST RETURN TIME DISTRIBUTION

In this section we will prove Theorems 3 and 4 which establish the limiting distribution of the first return time to Bowen balls and provide rates of convergence. We use the same notation as in the previous section.

$\tau(A)$ Be the period of A and as in [3] denote by $a_A = \mathbb{P}_A(\tau_A > \tau(A) + \Delta)$ the relative size of the set of long returns, where $\Delta < 1/\mu(A)$. Again we put $\tilde{B} = \tilde{B}_{\epsilon,n}(x)$ and $B = B_{\epsilon,n}(x)$ and let us first prove the following to lemmata.

We shall need the following result which compares the return times distributions for the dynamical ball B and the approximated set \tilde{B} .

Lemma 7. *There exists a constant C_8 so that*

$$\left| \mathbb{P}_B(\tau_B > \frac{t}{\lambda_B \mu(B)}) - \mathbb{P}_{\tilde{B}}(\tau_{\tilde{B}} > \frac{t}{\lambda_B \mu(\tilde{B})}) \right| \leq C_8 \frac{t\vartheta_n(\epsilon)}{\mu(B)}.$$

Proof. Let us first estimate the following term:

$$I = \left| \mathbb{P}_{\tilde{B}}(\tau_{\tilde{B}} > \frac{t}{\lambda_B \mu(\tilde{B})}) - \mathbb{P}_B(\tau_{\tilde{B}} > \frac{t}{\lambda_B \mu(\tilde{B})}) \right|$$

which is split into two parts $I \leq I_1 + I_2$. For the first part we obtain by Lemma 5

$$I_1 = \frac{1}{\mu(B)} \left| \mathbb{P}(\{\tau_{\tilde{B}} > \frac{t}{\lambda_B \mu(\tilde{B})}\} \cap \tilde{B}) - \mathbb{P}(\{\tau_{\tilde{B}} > \frac{t}{\lambda_B \mu(\tilde{B})}\} \cap B) \right| \leq \frac{\mu(\partial \tilde{B}_{\epsilon,n}(x))}{\mu(B)} \leq \frac{\vartheta_n(\epsilon)}{s}.$$

The second part is by (2)

$$I_2 = \mathbb{P} \left(\left\{ \tau_{\tilde{B}} > \frac{t}{\lambda_B \mu(\tilde{B})} \right\} \cap B \right) \left| \frac{1}{\mu(\tilde{B})} - \frac{1}{\mu(B)} \right| \leq \frac{\vartheta_n(\epsilon)}{s\mu(\tilde{B})}.$$

Hence

$$I \leq 2 \frac{\vartheta_n(\epsilon)}{s\mu(\tilde{B})}.$$

Let us next estimate the term

$$II = \left| \mathbb{P}_B(\tau_B > \frac{t}{\lambda_B \mu(B)}) - \mathbb{P}_B(\tau_{\tilde{B}} > \frac{t}{\lambda_B \mu(\tilde{B})}) \right|,$$

which again splits into two parts $II = II_1 + II_2$ as follows. The first part is

$$\begin{aligned} II_1 &= \left| \mathbb{P}_B(\tau_B > \frac{t}{\lambda_B \mu(B)}) - \mathbb{P}_B(\tau_B > \frac{t}{\lambda_B \mu(\tilde{B})}) \right| \\ &\leq \mathbb{P}_B\left(\frac{t}{\lambda_B \mu(B)} < t < \frac{t}{\lambda_B \mu(\tilde{B})}\right) \\ &\leq \frac{1}{\mu(B)} \left(\frac{t}{\lambda_B \mu(\tilde{B})} - \frac{t}{\lambda_B \mu(B)} \right) \mu(B) \\ &\leq \vartheta_n(\epsilon) \frac{t}{s \lambda_B \mu(\tilde{B})} \end{aligned}$$

by (2). For the second part we obtain

$$II_2 = \left| \mathbb{P}_B(\tau_B > \frac{t}{\lambda_B \mu(\tilde{B})}) - \mathbb{P}_B(\tau_{\tilde{B}} > \frac{t}{\lambda_B \mu(\tilde{B})}) \right| \leq \mathbb{P}_B(\tau_{B \setminus \tilde{B}} < \frac{t}{\lambda_B \mu(\tilde{B})}) \leq \frac{t}{\lambda_B} \frac{\mu(B \setminus \tilde{B})}{\mu(B) \mu(\tilde{B})}$$

and therefore by Lemma 5

$$II_2 \leq \frac{t}{\lambda_B} \frac{\mu(\partial \tilde{B}_{\epsilon, n}(x))}{\mu(B) \mu(\tilde{B})} \leq \frac{t \vartheta_n(\epsilon)}{s \lambda_B \mu(\tilde{B})}.$$

Finally we obtain for some constant C_8 that

$$I + II_1 + II_2 \leq C_8 \frac{t \vartheta_n(\epsilon)}{\mu(B)}$$

where we used that $s \lambda_B \geq C_7$ by Lemma 4 and $\frac{\mu(B)}{\mu(\tilde{B})} = \mathcal{O}(1)$. □

The following lemma is similar to Lemma 2.

Lemma 8. *For all Δ, f, g such that $f \geq \Delta > N(n)$, $g \geq \Delta + \tau(\tilde{B})$ we have*

$$\left| \mathbb{P}_{\tilde{B}}(\tau_{\tilde{B}} > f + g) - \mathbb{P}_{\tilde{B}}(\tau_{\tilde{B}} > g) \mathbb{P}(\tau_{\tilde{B}} > f) \right| \leq 2\Delta \mu(\tilde{B}_{\epsilon, n}(x)) + 2 \frac{\alpha(\Delta - N(n))}{\mu(\tilde{B})}$$

Proof. We proceed as in the proof of Lemma 2 to write the left-hand-side as $I + II + III$. The only difference is in I:

$$\begin{aligned} I &= |\mathbb{P}_{\tilde{B}}(\tau_{\tilde{B}} > f + g) - \mathbb{P}_{\tilde{B}}(\tau_{\tilde{B}} > g \cap \tau_{\tilde{B}} \circ T^{g+\Delta} > f - \Delta)| \\ &\leq \mathbb{P}_{\tilde{B}}(\tau_{\tilde{B}} \circ T^g \leq \Delta) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\mu(\tilde{B})} \mathbb{P}(\tilde{B} \cap \{\tau_{\tilde{B}} \circ T^g \leq \Delta\}) \\
&\leq \mathbb{P}(\tau_{\tilde{B}} \leq \Delta) + \frac{\alpha(\Delta - N(n))}{\mu(\tilde{B})} \leq \Delta\mu(\tilde{B}) + \frac{\alpha(\Delta - N(n))}{\mu(\tilde{B})}
\end{aligned}$$

by the α -mixing property. The estimates of the terms II and III are identical to the proof of Lemma 2. \square

Proof of Theorem 3. We will first show that $\mathbb{P}_{\tilde{B}}(\tau_{\tilde{B}} > \frac{t}{\lambda_B \mu(\tilde{B})})$ satisfies the exponential law.

For all $t > (\tau(\tilde{B}) + 2\Delta)\lambda_B \mu(\tilde{B})$, let $u = \frac{t}{\lambda_B \mu(\tilde{B})}$ we have

$$\begin{aligned}
&|\mathbb{P}_{\tilde{B}}(\tau_{\tilde{B}} > u) - \mathbb{P}_{\tilde{B}}(\tau_{\tilde{B}} > \tau(\tilde{B}) + \Delta)e^{-t}| \\
&\leq |\mathbb{P}_{\tilde{B}}(\tau_{\tilde{B}} > u) - \mathbb{P}_{\tilde{B}}(\tau_{\tilde{B}} > \tau(\tilde{B}) + \Delta)\mathbb{P}(\tau_{\tilde{B}} > u - (\tau(\tilde{B}) + \Delta))| \\
&\quad + |\mathbb{P}_{\tilde{B}}(\tau_{\tilde{B}} > \tau(\tilde{B}) + \Delta)\mathbb{P}(\tau_{\tilde{B}} > u - (\tau(\tilde{B}) + \Delta)) - \mathbb{P}_{\tilde{B}}(\tau_{\tilde{B}} > \tau(\tilde{B}) + \Delta)e^{-t}| \\
&= I + II
\end{aligned}$$

where

$$I \leq 2\Delta\mu(\tilde{B}) + 2\frac{\alpha(\Delta - N(n))}{\mu(\tilde{B})}$$

by Lemma 8. For II we have

$$\begin{aligned}
II &= a_B |\mathbb{P}(\tau_{\tilde{B}} > \frac{t}{\lambda_B \mu(\tilde{B})} - (\tau(\tilde{B}) + \Delta)) - e^{-t}| \\
&\leq a_B |\mathbb{P}(\tau_{\tilde{B}} > \frac{t}{\lambda_B \mu(\tilde{B})} - (\tau(\tilde{B}) + \Delta)) - e^{-t+(\tau(\tilde{B})+\Delta)\lambda_B \mu(\tilde{B})}| + a_B |e^{-t} - e^{-t+(\tau(\tilde{B})+\Delta)\lambda_B \mu(\tilde{B})}|.
\end{aligned}$$

To the first term we apply Theorem 1 with the parameter value $t' = t - (\tau(\tilde{B}) + \Delta)\lambda_B \mu(\tilde{B})$ and to the second term we apply the Mean Value Theorem. Hence $II \leq c_1 \mu(\tilde{B})^{\omega_1}$ for some $\omega_1 > 0$ from Theorem 1. This proves Theorem 3 for the set \tilde{B} . To prove the theorem for the set B we use Lemma 7 and put $N(n) = \mu(B)^{-\eta}$ for some $\eta \in (0, 1/2)$. Thus

$$\begin{aligned}
&\left| \mathbb{P}_B(\tau_B > \frac{t}{\lambda_B \mu(B)}) - \mathbb{P}_{\tilde{B}}(\tau_{\tilde{B}} > \frac{t}{\lambda_B \mu(\tilde{B})}) \right| \\
&\leq C_8 \frac{t \vartheta_n(\epsilon)}{\mu(B)} \\
&= \frac{\mathcal{O}(t)ns}{|\log \gamma^N|^{5+\zeta} \mu(B)^2}
\end{aligned}$$

$$\begin{aligned}
&= \mathcal{O}(t)ns\mu(B)^{(5+\zeta)\eta-2} \\
&= \mathcal{O}(t)n\mu(B)^{(5+\zeta)\eta-2} \left(\mu(B)^{-\frac{1}{2+\kappa}} + \mu(B)^{-\eta} \right).
\end{aligned}$$

by (3) and as $s\lambda_B = \mathcal{O}(1)$ and $s = \alpha^{-1}(C'\mu(\tilde{B})) + N(n)$ by Lemma 4. A choice of η close to $\frac{1}{2}$ will achieve that $\omega_2 = (5 + \zeta)\eta - 2 - \max\{\frac{1}{2+\kappa}, \eta\}$ is positive. Now put $\omega = \min\{\omega_1, \omega_2\}$. \square

Proof of Theorem 4. The first part of the proof is identical to the proof of Theorem 3. For the second part we get different estimates as $\text{diam}(\mathcal{A}^n) = \mathcal{O}(n^{-a})$ for some a . To prove the theorem for the set B we use Lemma 7 and put $N(n) = \mu(B)^{-\eta}$ for some $\eta \in (0, 1/2)$. Thus

$$\begin{aligned}
&\left| \mathbb{P}_B(\tau_B > \frac{t}{\lambda_B \mu(B)}) - \mathbb{P}_{\tilde{B}}(\tau_{\tilde{B}} > \frac{t}{\lambda_B \mu(\tilde{B})}) \right| \\
&\leq C_8 \frac{t\vartheta_n(\epsilon)}{\mu(B)} \\
&= \frac{\mathcal{O}(t)ns}{N(n)^{a\zeta}\mu(B)^2} \\
&= \mathcal{O}(t)n\mu(B)^{a\zeta\eta-2} \left(\mu(B)^{-\frac{1}{2+\kappa}} + \mu(B)^{-\eta} \right).
\end{aligned}$$

by (3). A choice of η close to $\frac{1}{2}$ will achieve that $\omega_2 = a\zeta\eta - 2 - \max\{\frac{1}{2+\kappa}, \eta\}$ is positive. Now put $\omega = \min\{\omega_1, \omega_2\}$ (ω_1 from the proof of Theorem 3). \square

5. MAPS WITH DECAYING CORRELATIONS

In this section we show how the results of the previous section can be applied to dynamical systems that can be modelled by a Markov tower as Young constructed in [23, 24].

We assume that T is a differentiable map on the manifold X . Then one assumes there is a subset $\Omega_0 \subset X$ with the following properties:

(i) Ω_0 is partitioned into disjoint sets $\Omega_{0,i}$, $i = 1, 2, \dots$ and there is a *return time function* $R : \Omega_0 \rightarrow \mathbb{N}$, constant on the partition elements $\Omega_{0,i}$, such that T^R maps $\Omega_{0,i}$ bijectively to the entire set Ω_0 . We write $R_i = R|_{\Omega_{0,i}}$. Moreover, it is assumed that the $\Omega_{0,i}$ are rectangles, that is, if $\gamma^u(x)$ denotes the unstable leaf through $x \in \Omega_{0,i}$ and $\gamma^s(y)$ the stable leaf at $y \in \Omega_{0,i}$, then there is a unique intersection $\gamma^u(x) \cap \gamma^s(y)$ which also lies in $\Omega_{0,i}$. It is also assumed that the $\Omega_{0,i}$ satisfy the Markov property. If γ^u and $\hat{\gamma}^u$ are two unstable leaves (in some $\Omega_{i,0}$), then the holonomy $\Theta : \gamma^u \rightarrow \hat{\gamma}^u$ is given by $\Theta(x) = \hat{\gamma}^u \cap \gamma^s(x)$, $x \in \gamma^u$.

(ii) For $j = 0, 1, \dots, R_i - 1$ put $\Omega_{j,i} = \{(x, j) : x \in \Omega_{0,i}\}$ and define $\Omega = \bigcup_{i=1}^{\infty} \bigcup_{j=0}^{R_i-1} \Omega_{j,i}$. Ω is the *Markov tower* for the map T . It has the associated partition $\mathcal{A} = \{\Omega_{j,i} :$

$0 \leq j < R_i, i = 1, 2, \dots\}$ which typically is countably infinite. The map $F : \Omega \rightarrow \Omega$ is given by

$$F(x, j) = \begin{cases} (x, j+1) & \text{if } j < R_i - 1 \\ (\hat{F}x, 0) & \text{if } j = R_i - 1 \end{cases}$$

where we put $\hat{F} = F^R$ for the induced map on Ω_0 .

(iii) The *separation function* $s(x, y)$, $x, y \in \Omega_0$, is defined as the largest positive n so that $(T^R)^j x$ and $(T^R)^j y$ lie in the same sub-partition elements for $0 \leq j < n$. That is $(T^R)^j x, (T^R)^j y \in \Omega_{0, i_j}$ for some $i_0, i_1, \dots, n-1$. We extend the separation function to all of Ω by putting $s(x, y) = s(T^{R-j}x, T^{R-j}y)$ for $s, y \in \Omega_{j, i}$.

(iv) Let ν be a finite given ‘reference’ measure on Ω and let ν_{γ^u} be the conditional measure on the unstable leaves. We assume that the Jacobian $JF = \frac{d(F_*^{-1}\nu_{\gamma^u})}{d\nu_{\gamma^u}}$ is Hölder continuous in the following sense: There exists a $\gamma \in (0, 1)$ so that

$$\left| \frac{JF^R x}{JF^R y} - 1 \right| \leq \text{const} \gamma^{s(\hat{F}x, \hat{F}y)}$$

for all $x, y \in \Omega_{0, i}$, $i = 1, 2, \dots$.

If the return time R is integrable with respect to ν then by [24] Theorem 1 there exists an F -invariant probability measure μ (SRB measure) on Ω which is absolutely continuous with respect to ν .

Theorem 6. *Let μ be the SRB measure for a differentiable map T on a manifold X . If the return times decay at least polynomially with power $\lambda > 5 + \sqrt{15}$, then*

$$\lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} \mathbb{P} \left(\tau_{B_{\epsilon, n}(x)} \geq \frac{t}{\lambda_{B_{\epsilon, n}(x)} \mu(B_{\epsilon, n}(x))} \right) = e^{-t}$$

for $t > 0$ and almost every $x \in X$.

For each $n \in \mathbb{N}$ the elements of the n th join $\mathcal{A}^n = \bigvee_{i=0}^{n-1} T^{-i} \mathcal{A}$ of the partition $\mathcal{A} = \{\Omega_{i, j}\}$ are called n -cylinders and form a new partition of Ω , a refinement of the original partition. The σ -algebra generated by all n -cylinders \mathcal{A}^ℓ , for all $\ell \geq 1$, is the σ -algebra of the system (Ω, μ) .

Lemma 9. *The invariant measure μ is α -mixing with respect to the partition \mathcal{A} , where $\alpha(k) \sim p(k)$.*

Proof. Denote by \mathcal{C}_γ the space of Hölder continuous functions φ on Ω for which $|\varphi(x) - \varphi(y)| \leq C_\varphi \gamma^{s(x, y)}$. If C_φ is smallest then $\|\varphi\|_\gamma = |\varphi|_\infty + C_\varphi$ defines a norm and $\mathcal{C}_\gamma = \{\varphi : \|\varphi\|_\gamma < \infty\}$. Let $\mathcal{L} : \mathcal{C}_\gamma \rightarrow \mathcal{C}_\gamma$ be the transfer operator defined by $\mathcal{L}\varphi(x) = \sum_{x' \in T^{-1}x} \frac{\varphi(x')}{JT(x')}$, $\varphi \in \mathcal{C}_\gamma$. Then ν is a fixed point of its adjoint, i.e. $\mathcal{L}^* \nu = \nu$

and $h = \frac{d\mu}{d\nu} = \lim_{n \rightarrow \infty} \mathcal{L}^n \lambda$ is Hölder continuous, where λ can be any initial density distribution in \mathcal{C}_γ . In fact, by [24] Theorem 2(II) one has

$$(4) \quad \|\mathcal{L}^k \lambda - h\|_{\mathcal{L}^1} \leq p(k) \|\lambda\|_\gamma$$

where the ‘decay function’ $p(k) = \mathcal{O}(k^{-\beta})$ if the return times decay polynomially with power β , that is if $\nu(R > j) \leq \text{const.} j^{-\beta}$. If the return times decay exponentially, i.e. if $\nu(R > j) \leq \text{const.} \vartheta^j$ for some $\vartheta \in (0, 1)$, then there is a $\tilde{\vartheta} \in (0, 1)$ so that $p(k) \leq \text{const.} \tilde{\vartheta}^k$.

As in the proof of [24] Theorem 3 we put $\lambda = \mathcal{L}^n h \chi_A$ which is a strictly positive function. Then $\eta = \frac{\lambda}{\mu(A)}$ is a density function as $\nu(\lambda) = \nu(\mathcal{L}^n h \chi_A) = \nu(h \chi_A) = \mu(A)$. Since by [12] there exists a constant c_1 so that $\|\mathcal{L}^n \chi_A\|_\gamma \leq c_1$ for all $A \in \sigma(\mathcal{A}^n)$ and n we see that $\|\lambda\|_\gamma \leq c_1$ uniformly in n and $A \in \sigma(\mathcal{A}^n)$. Hence

$$\begin{aligned} \mu(A \cap T^{-k-n} B) - \mu(A)\mu(B) &= \nu(h \chi_A (\chi_B \circ T^{k+n})) - \nu(h \chi_A) \nu(h \chi_B) \\ &= \mu(A) (\nu(\chi_B \mathcal{L}^k \eta) - \nu(h \chi_B)) \\ &= \mu(A) \int \chi_B (\mathcal{L}^k \eta - h) d\nu \\ &= \int_B (\mathcal{L}^k \lambda - \mu(A)h) d\nu. \end{aligned}$$

Using the estimates from the \mathcal{L}^1 -convergence of $\mathcal{L}^k \eta - h$ from (4) yields

$$\begin{aligned} |\mu(A \cap T^{-k-n} B) - \mu(A)\mu(B)| &\leq \mu(A) \int \chi_B |\mathcal{L}^k \lambda - h| d\nu \\ &\leq \mu(A) c_1 \|\eta\|_\gamma p(k) \\ &\leq c_3 p(k) \end{aligned}$$

as $\|\eta\|_\gamma = \frac{1}{\mu(A)} \|\lambda\|_\gamma \leq \frac{C_3}{\mu(A)}$. In particular we can write

$$|\mu(A \cap T^{-k-n} B) - \mu(A)\mu(B)| \leq \alpha(k)$$

for all $A \in \sigma(\mathcal{A}^n)$, $B \in \sigma(\bigcup_{j \geq 1} \mathcal{A}^j)$, where $\alpha(k) = c_3 p(k)$. \square

Lemma 10. *Let $\xi < \frac{\lambda}{2} - 1$. Then there exists an ϵ_0 so that for every $\delta < \epsilon_0$ there exists a set $\mathcal{U}_\delta \subset X$, of measure $\mathcal{O}(|\log \delta|^{-\xi})$ so that $\varphi(\epsilon, \delta, x) = \mathcal{O}(|\log \delta|^{-\xi})$ uniformly in $x \notin \mathcal{U}_\delta$.*

Proof. It was shown in [13] Proposition 6.1 that for all w large enough there exists a set $\mathcal{U} \subset X$ such that $\mu(\mathcal{U}) = \mathcal{O}((w|\log \epsilon|)^{-\xi})$ and $\varphi(\epsilon, \epsilon^w, x) = \mathcal{O}((w|\log \epsilon|)^{-\xi})$ uniformly in $x \notin \mathcal{U}$ where ξ is any number less than $\frac{\lambda}{2} - 1$. Hence there exists an $\epsilon_0 > 0$ so that we can write $\delta = \epsilon^w$ with w large enough (larger than $\frac{2}{u}(D+1) - 1$ where D is the dimension of the manifold X and u is the dimension of the unstable leaves) for all $\delta < \epsilon_0$. Since $\log \delta = w \log \epsilon$ we obtain the statement of the lemma. \square

Let us denote by $\tilde{\Omega}_{j,i}$ the principal parts of $\Omega_{j,i}$. For integers N, m ($N \gg m$) we put $\tilde{\Omega}_{j,i} = \{x \in \Omega_{j,i} : R(\hat{F}^j x) \leq s \ \forall j = 0, \dots, [N/m]\}$. In this way we pick out the return times that are not too long. In particular $\tilde{\Omega}_{0,i} = \emptyset$ if $R_i > m$. Let us put $\tilde{\Omega} = \bigcup_i \bigcup_{j=0}^{R_i-1} \tilde{\Omega}_{j,i}$ (disjoint unions).

We also define $\omega(m) = \sqrt{\sum_{i: R_i > m} R_i \nu(\Omega_{0,i})}$ and note that $\omega(m) = \mathcal{O}(m^{-\frac{\lambda-1}{2}})$.

Lemma 11. [13] *There exists a constant C_9 and for $N, n, m \geq 1$ ($N > n, m$) there exist sets $\mathcal{V}_{N,m} \subset M$ such that the non-principal part contributions are estimated as*

$$\mu(\mathcal{B} \cap (\Omega \setminus \tilde{\Omega})) < \sqrt{n+2} \omega(m) \mu(B_{\epsilon,n})$$

for any $\mathcal{B} \subset B_{\epsilon,n}(x)$ and $x \notin \mathcal{V}_{N,m}$ where

$$\mu(\mathcal{V}_{N,m}) \leq C_9 \sqrt{n+2} \omega(m).$$

Proof of Theorem 6. In order to apply Theorem 5 to $B_{\epsilon,n}(x) \cap \tilde{\Omega}$ we will pick $\eta \in (0, \frac{1}{2})$ below and put $N(n) = [\mu(B_{\epsilon,n}(x))^{-\eta}]$. We then choose $m = N^\alpha$ for some $\alpha \in (\frac{1}{\lambda-1}, 1)$ (see estimate of \mathcal{F} below). Then according to Lemma 11 $\text{diam}(A) \leq \gamma \frac{N}{m}$ for some $\gamma < 1$ for all n -cylinders A which belong to $\tilde{\Omega}$. As in the proof of Theorem 1 we then conclude that

$$\varphi(\epsilon, \gamma_{N(n)-k}, T^k x) \leq \frac{c_1 m^\xi}{N(n)^\xi} \leq c_1 m^\xi \mu(B)^{\xi\eta}$$

(for some constant c_1) provided $T^k x \notin \mathcal{U}_{\gamma_{N(n)-k}}$ for $k = 0, \dots, n-1$. Since $s = \alpha^{-1}(C' \mu(B)) + N(n) \leq c_2 \mu(B)^{-\frac{1}{\lambda}} + \mu(B)^{-\eta}$, we obtain

$$\frac{ns \cdot \varphi(\epsilon, \gamma_{N(n)-k}, T^k x)}{\mu(B)} \leq c_3 n \cdot \mu(B)^{\eta\xi-1-\eta\alpha\xi} \left(\mu(B)^{-\frac{1}{\lambda}} + \mu(B)^{-\eta} \right).$$

The RHS converges to zero if $\eta\xi - 1 - \eta\alpha\xi - \max\{\frac{1}{\lambda}, \eta\}$ is positive. To satisfy Lemma 10 it is required that $\xi < \frac{\lambda}{2} - 1$. Then can choose $\alpha \in (\frac{1}{\lambda-1}, 1)$ in such a way that the above expression is positive for an $\eta < \frac{1}{2}$ close to $\frac{1}{2}$. This can be done if $\lambda > 5 + \sqrt{15}$.

We now proceed as in the proof Theorem 1 to estimate the contribution to the error made by $(B \setminus \tilde{B}) \cap \tilde{\Omega}$. For the portion that lies in $\Omega \setminus \tilde{\Omega}$ we use Lemma 11 and thus obtain combining the two contributions:

$$\left| \mathbb{P}(\tau_{B_{\epsilon,n}(x)} > \frac{t}{\lambda_{B_{\epsilon,n}(x)} \mu(B_{\epsilon,n}(x))}) - e^{-t} \right| \leq c_4 \left(t \mu(B)^a + \mu(B)^b + \sqrt{N} \omega(m) \mu(B) \right)$$

provided x does not lie in the forbidden set $\mathcal{F} = \mathcal{V}_{N,m} \cup \bigcup_{k=0}^{n-1} T^{-k} \mathcal{U}_{\gamma_{N(n)-k}}$ whose measure is by Lemmata 11 and 10 bounded by

$$\mu(\mathcal{F}) \leq c_5 \left(\sqrt{N+2} \omega(m) + n |\log \gamma_{N(n)}|^{-\xi} \right) \leq c_6 \left(\sqrt{N} m^{-\frac{\lambda-1}{2}} + n N^{-\xi} m^\xi \right)$$

which goes to zero as $n \rightarrow \infty$ since $m = N^\alpha$ and $\frac{1}{\lambda-1} < \alpha < 1$. Thus $\mu(\mathcal{F}) \rightarrow 0$ as $n \rightarrow \infty$ and therefore $\mathbb{P}(\tau_{B_{\epsilon,n}(x)} > \frac{t}{\lambda_{B_{\epsilon,n}(x)}\mu(B_{\epsilon,n}(x))}) \rightarrow e^{-t}$ as $n \rightarrow \infty$, $\epsilon \rightarrow 0$ for every $x \notin \liminf_{n \rightarrow \infty, \epsilon \rightarrow 0} \mathcal{F}_{\epsilon,n}$.

□

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